A Jump Diffusion Option Pricing Model

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In this white paper we will build a jump diffusion model for option price. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to price ABC Company call options given the following go-forward model assumptions...

Table 1: Model Parameters

Symbol	Description	Amount
S_0	Stock price at time zero ($)	100.00
X_t	Call option exercise price ($)	100.00
α	Risk-free interest rate (%)	3.00
ϕ	Dividend yield (%)	5.00
μ	Expected return mean (%)	15.00
σ	Expected return volatility (%)	25.00
ω	Jump size mean (%)	4.00
υ	Jump size volatility (%)	15.00
λ	Average number of annual jumps (#)	3.25
t	Time in years (#)	3.00

Our task is to answer the following question...

Question: What is value of the call option at time zero?

Equation For Stock Price Under The Actual Probability Measure P

We will define the variable $S(k)_t$ to be conditional random stock price at time t under the actual probability Measure P, the variable μ to be expected annualized total return including jumps, the variable z to be a normally-distributed random variable, and the variable k to be a Poisson-distributed random variable. Using the data in Table 1 above the equation for conditional random stock price at time t given k jumps over the time interval $[0,t]$ is... [4]

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{(\mu - \phi - \tfrac{1}{2}\,\sigma_k^2)\,t - \lambda\,\omega\,t + k\,\ln(1+\omega) + \sigma_k\sqrt{t}\,z\right\} \text{ ...where... } \sigma_k = \sqrt{\sigma^2\,t + \frac{\upsilon^2 k}{t}\,t}$$

$$\text{...and... } z \sim N\left[0,1\right\} \text{ ...and... } k \sim P\left[\lambda\,t, \lambda\,t\right] \text{ ...and... } \rho_{z,k} = 0 \tag{1}$$

Using Equation (1) above we can separate the jump and diffusion processes and rewrite that equation for conditional random stock price as...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{k\,\ln(1+\omega) - \lambda\,\omega\,t\right\} \mathrm{Exp}\left\{(\mu - \phi - \tfrac{1}{2}\,\sigma_k^2)\,t + \sigma_k\sqrt{t}\,z\right\} \tag{2}$$

Using Equation (2) above the equation for expected conditional stock price at time t given k jumps over the time interval $[0,t]$ under the actual probability Measure P is... [4]

$$\mathbb{E}^P\left[S(k)_t\right] = S_0 \,\mathrm{Exp}\left\{k\,\ln(1+\omega) - \lambda\,\omega\,t\right\} \mathrm{Exp}\left\{(\mu - \phi)\,t\right\} \tag{3}$$

The equation for the probability of k jumps over the time interval $[0,t]$ is... [2]

$$\mathrm{Prob}\bigg[k\bigg] = \frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} \tag{4}$$

Using Equations (3) and (4) above the equation for expected unconditional stock price at time t under the actual probability Measure P is... [4]

$$\mathbb{E}^P\bigg[S_t\bigg] = \sum_{k=0}^{\infty}\mathrm{Prob}\bigg[k\bigg]\,\mathbb{E}^P\bigg[S(k)_t\bigg] = S_0\,\mathrm{Exp}\bigg\{(\mu-\phi)\,t\bigg\} \tag{5}$$

Equation For Stock Price Under The Risk-Neutral Probability Measure Q

We will define the variable $\bar{S}(k)_t$ to be conditional random stock price at time t under the risk-neutral probability Measure Q and the variable α to be the risk-free interest rate. Using Equation (2) above the equation for conditional random stock price at time t given k jumps over the time interval $[0,t]$ is...

$$\bar{S}(k)_t = S_0\,\mathrm{Exp}\bigg\{k\,\ln(1+\omega)-\lambda\,\omega\,t\bigg\}\,\mathrm{Exp}\bigg\{(\alpha-\phi-\tfrac{1}{2}\,\sigma_k^2)\,t+\sigma_k\sqrt{t}\,z\bigg\} \tag{6}$$

Using Equations (3) and (6) above the equation for expected conditional stock price at time t given k jumps over the time interval $[0,t]$ under the risk-neutral probability Measure Q is...

$$\mathbb{E}^Q\bigg[\bar{S}(k)_t\bigg] = S_0\,\mathrm{Exp}\bigg\{k\,\ln(1+\omega)-\lambda\,\omega\,t\bigg\}\,\mathrm{Exp}\bigg\{(\alpha-\phi)\,t\bigg\} \tag{7}$$

Using Equations (5) and (7) above the equation for expected unconditional stock price at time t under the risk-neutral probability Measure Q is...

$$\mathbb{E}^Q\bigg[\bar{S}_t\bigg] = \sum_{k=0}^{\infty}\mathrm{Prob}\bigg[k\bigg]\,\mathbb{E}^Q\bigg[\bar{S}(k)_t\bigg] = S_0\,\mathrm{Exp}\bigg\{(\alpha-\phi)\,t\bigg\} \tag{8}$$

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We will define the variable $S(k)_0$ to be spot price at time zero given k jumps over the time interval $[0,t]$. Using Equation (3) above the equation for conditional spot price is...

$$S(k)_0 = S_0\,\mathrm{Exp}\bigg\{k\,\ln(1+\omega)-\lambda\,\omega\,t\bigg\} \tag{9}$$

We will define the variable $C(k)_0$ to be conditional call option value at time zero given k jumps over the time interval $[0,t]$. Using Equation (9) above and the data in Table 1 above the equation for conditional call option value at time zero is... [1]

$$C(k)_0 = \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\mathrm{Max}\bigg[S(k)_0\,\mathrm{Exp}\bigg\{\bigg(\alpha-\phi-\frac{1}{2}\,\sigma^2\bigg)t+\sigma\,\sqrt{t}\,z\bigg\}-X_t,0\bigg]\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta z$$

$$\text{...where... } z\sim N\bigg[0,1\bigg\}\ \text{ ...and... } k\sim P\bigg[\lambda\,t,\lambda\,t\bigg] \tag{10}$$

We will define the function $CNDF(z)$ to be the cumulative normal distribution function where the random variable z is normally-distributed with mean zero and variance one. The solution to Equation (10) above is... [1]

$$C(k)_0 = S(k)_0\,\mathrm{Exp}\bigg\{-\phi\,t\bigg\}\,\mathrm{CNDF}\bigg[-a+\sigma\,\sqrt{t}\bigg]-X_t\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\mathrm{CNDF}\bigg[-a\bigg] \tag{11}$$

Using Equations (4) and (11) above the equation for unconditional call option value at time zero is...

$$\begin{aligned} C_0 &= \sum_{k=0}^{\infty}\mathrm{Prob}\bigg[k\bigg]\,C(k)_0 \\ &= \sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,S(k)_0\,\mathrm{Exp}\bigg\{-\phi\,t\bigg\}\,\mathrm{CNDF}\bigg[-a+\sigma\,\sqrt{t}\bigg]-X_t\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\mathrm{CNDF}\bigg[-a\bigg] \end{aligned} \tag{12}$$

The Answer To Our Hypothetical Problem

Question: What is value of the call option at time zero?

The answer to the question is $20.09 per the following table...

Table 2: Call Option Value

A Jumps k	B Jump Prob	C BS Spot Price	D BS Volatility	E BS Call Price	F Prob Wt Price
0	0.00006	67.71	0.25000	2.41	0.0001
1	0.00057	70.41	0.26458	3.39	0.0019
2	0.00277	73.23	0.27839	4.55	0.0126
3	0.00901	76.16	0.29155	5.89	0.0530
4	0.02195	79.21	0.30414	7.39	0.1622
5	0.04280	82.37	0.31623	9.06	0.3876
6	0.06955	85.67	0.32787	10.89	0.7575
7	0.09688	89.10	0.33912	12.89	1.2490
8	0.11807	92.66	0.35000	15.06	1.7780
9	0.12791	96.37	0.36056	17.39	2.2248
10	0.12471	100.22	0.37081	19.90	2.4813
11	0.11054	104.23	0.38079	22.57	2.4948
12	0.08981	108.40	0.39051	25.42	2.2827
13	0.06736	112.73	0.40000	28.44	1.9157
14	0.04691	117.24	0.40927	31.64	1.4844
15	0.03049	121.93	0.41833	35.03	1.0681
16	0.01858	126.81	0.42720	38.60	0.7173
17	0.01066	131.88	0.43589	42.37	0.4516
18	0.00577	137.16	0.44441	46.34	0.2675
19	0.00296	142.65	0.45277	50.51	0.1496
20	0.00144	148.35	0.46098	54.89	0.0793
21	0.00067	154.29	0.46904	59.48	0.0399
22	0.00030	160.46	0.47697	64.30	0.0191
23	0.00013	166.88	0.48477	69.35	0.0087
24	0.00005	173.55	0.49244	74.64	0.0038
25	0.00002	180.49	0.50000	80.17	0.0016
26	0.00001	187.71	0.50744	85.96	0.0006
27	0.00000	195.22	0.51478	92.01	0.0002
28	0.00000	203.03	0.52202	98.33	0.0001
29	0.00000	211.15	0.52915	104.93	0.0000
30	0.00000	219.60	0.53619	111.83	0.0000
Total	1.00000	–	–	–	20.0933

Column B - Jump probability Equation (4)
Column C - BS spot price Equation (9)
Column D - BS volatility Equation (1)
Column E - BS conditional call price Equation (11)
Column F - BS unconditional call price Equation (12)

References

[1] Gary Schurman, *Solving The Black-Scholes Option Pricing Integral,* August, 2019.

[2] Gary Schurman, *The Poisson Distribution,* June, 2012.

[3] Gary Schurman, *A Jump Diffussion Model For Stock Price,* March, 2021.

[4] Gary Schurman, *Combining Diffusion And Jump Size Variances,* March, 2021.